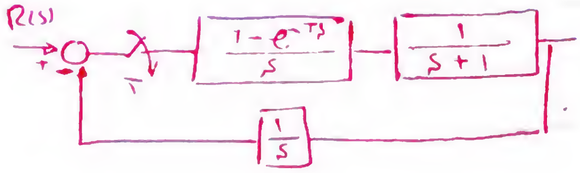
Sheet 3

1 For the given System

Required : output $c(k)$ for unit step input

Discuss System Stability



Sol^n

$$G(s) = \frac{1-e^{-Ts}}{s(s+1)} \quad , \quad GH(s) = \frac{1-e^{-Ts}}{s^2(s+1)}$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+\overline{GH}(z)}$$

$$\rightarrow G(z) = Z\left[\frac{1-e^{-Ts}}{s(s+1)}\right]_{T=1} = (1-z^{-1})\, Z\left[\frac{1}{s} - \frac{1}{s+1}\right] = (1-z^{-1})\, Z\left[1 - e^{-t}\right]$$

$$= (1-z^{-1})\left[\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right] = 1 - \frac{z-1}{z-0.368} = \boxed{\frac{0.632}{z-0.368}}$$

$$\rightarrow \overline{GH}(z) = Z\left[\frac{1-e^{-Ts}}{s^2(s+1)}\right] = (1-z^{-1})\, Z\left[\frac{1}{s^2(s+1)}\right] = (1-z^{-1})\, Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$= (1-z^{-1})\, Z\left[t - 1 + e^{-t}\right] = \frac{z-1}{z}\left[\frac{z}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{z-e^{-1}}\right]$$

$$= \frac{1}{z-1} - 1 + \frac{z-1}{z-0.368} = \frac{(z-0.368) - (z-1)(z-0.368) + (z-1)^2}{(z-1)(z-0.368)}$$

$$= \frac{z - 0.368 - \cancel{z^2} + 1.368z - 0.368 + \cancel{z^2} - 2z + 1}{(z-1)(z-0.368)} = \boxed{\frac{0.368z + 0.264}{(z-1)(z-0.368)}}$$

$$\therefore \frac{C(z)}{R(z)} = \frac{\frac{0.632}{z-0.368}}{1 + \frac{0.368z + 0.264}{(z-1)(z-0.368)}} = \frac{0.632(z-1)}{(z-1)(z-0.368) + (0.368z + 0.264)}$$

$$\frac{C(z)}{R(z)} = \boxed{\frac{0.632(z-1)}{z^2 - z + 0.632}}$$

For unit step Input $R(z) = \frac{z}{z-1}$

$$\therefore C(z) = \frac{0.632(\cancel{z-1})}{z^2 - z + 0.632} \cdot \frac{z}{\cancel{z-1}} = \frac{0.632\, z}{z^2 - z + 0.632}$$

Remember

$$Z\left[a^k f(k)\right] = F(z)\Big|_{z=\frac{z}{a}} \longrightarrow Z\left[a^k \sin\omega t\right] = \frac{\frac{z}{a}\sin\omega T}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\omega T + 1}$$

for $\underline{\underline{T=1}}$

$$= \frac{z a \sin(\omega T)}{z^2 - 2za\cos\omega T + a^2}$$

$$\because C(z) = \frac{0.632\, z}{z^2 - z + 0.632}$$ By analogy

$\therefore a^2 = 0.632 \longrightarrow a = 0.795$

$2a\cos\omega = 1 \longrightarrow \cos\omega = \frac{1}{2 * 0.795} = 0.629 \longrightarrow \omega = 51.03^\circ = 0.891$ rads

$\therefore a\sin\omega = 0.795 * 0.777 = 0.618$

$$\therefore C(z) = \underbrace{\frac{0.632}{0.618}}_{1.022} \cdot \frac{\overbrace{(0.618)}^{a\sin\omega}\, z}{z^2 - \underbrace{z}_{2a\cos\omega} + \underbrace{0.632}_{a^2}}$$ to match the form

$$\therefore \boxed{C(k) = 1.022\,(0.795)^k \sin(0.891\, k)}$$ #

→ System Stability

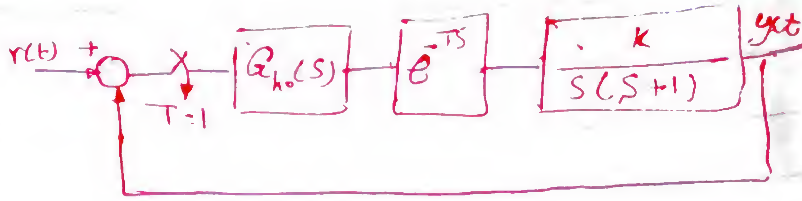
$$\frac{C(z)}{R(z)} = \frac{0.632(z-1)}{z^2 - z + 0.632}$$

c/c eqn: $z^2 - z + 0.632 = 0 \longrightarrow z_{1,2} = \frac{1 \pm \sqrt{1 - 4*0.632}}{2} = 0.5 \pm j0.618$

$\|z_{1,2}\| = \sqrt{0.5^2 + 0.618^2} = 0.632 < 1$

the poles located inside the unit circle ⟶ Stable System

For the given System
i) Determine the O.L.T.F
ii) what is the type of the System
iii) Calculate e_{ss} for unit ramp
iv) Calculate $y(\infty)$ for unit step
Assume $k = 0.1$



Sol^{n}

i) The O.L.T.F. $G(z) = Z\left[\frac{1-e^{-TS}}{S} \cdot e^{-TS} \cdot \frac{K}{S(S+1)}\right]$

$$= K(1-z^{-1})\, z^{-1}\, Z\left[\frac{1}{S^2(S+1)}\right]$$

$$= K\frac{(z-1)}{z^2} Z\left[\frac{1}{S^2} - \frac{1}{S} + \frac{1}{S+1}\right]$$

$$= K\frac{\cancel{(z-1)}}{z^{\cancel{2}}}\left[\frac{T\cancel{z}}{(z-1)^{\cancel{2}}} - \frac{\cancel{z}}{z-1} + \frac{\cancel{z}}{z-e^{-T}}\right] \quad T=1$$

$$= \frac{K}{z}\left[\frac{1}{z-1} - 1 + \frac{z-1}{z-0.368}\right]$$

$$= \frac{K}{z}\left[\frac{(z-0.368)-(z-1)(z-0.368)+(z-1)^2}{(z-1)(z-0.368)}\right]$$

$$= \frac{K}{z}\left[\frac{z-0.368-\cancel{z^2}+1.368z-0.368+\cancel{z^2}-2z+1}{(z-1)(z-0.368)}\right]$$

$$= \boxed{\frac{K(0.368z+0.264)}{z(z-1)(z-0.368)}}$$

ii) The System is type 1

Note: type of discrete System is the order of $(z-1)$ term

iii) For unit ramp Input $e_{ss} = \frac{1}{K_v}$

$$K_v = \frac{1}{T} \lim_{z \to 1} (1 - z^{-1}) G(z)$$

$$= \lim_{z \to 1} \frac{z-1}{z} \cdot \frac{K(0.368z + 0.264)}{z(z-1)(z-0.368)} = \frac{K(0.632)}{1-0.368} = K$$

$e_{ss} = \frac{1}{K}$ for $K = 0.1 \longrightarrow \boxed{e_{ss} = 10}$

iv) $$y(\infty) = \lim_{t \to \infty} y(t) = \lim_{z \to 1} \frac{(z-1)}{z} Y(z)$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{K(0.368z + 0.264)}{z(z-1)(z-0.368) + K(0.368z + 0.264)}$$

$$= \frac{K(0.368z + 0.264)}{z^3 - 1.368z^2 + 0.368(1+K)z + 0.264K}$$ C.L.T.F

for unit step Input $\longrightarrow R(z) = \frac{z}{z-1}$

$$Y(z) = \frac{K(0.368z + 0.264)}{z^3 - 1.368z^2 + 0.368(1+K)z + 0.264K} \cdot \frac{z}{z-1}$$ (Type 1)

$$y_{ss} = y(\infty) = \lim_{z \to 1} \frac{\cancel{(z-1)} * K(0.368z + 0.264) * \cancel{z}}{\cancel{z}\,(z^3 - 1.368z^2 + 0.368(1+K)z + 0.264K)\cancel{(z-1)}}$$

$$= \frac{0.632 * K}{1 - 1.368 + 0.368 + 0.368K + 0.264K} = \frac{0.632K}{0.632K} = 1$$

Another Solution:

- For unit step input $e_{ss} = \frac{1}{1+K_p}$, $K_p = \lim_{z \to 1} G(z) = \infty$

$\therefore e_{ss} = \text{zero}$

$\because e_{ss} = r_{ss} - y_{ss}$

$y_{ss} = r_{ss} - e_{ss} = 1 - 0 = 1$